THE TEACHING OF TRADITIONAL STANDARD ALGORITHMS FOR THE FOUR ARITHMETIC OPERATIONS VERSUS THE USE OF PUPILS' OWN METHODS

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Abstract: *In this article I discuss some reasons why it might be advantageous to let pupils use their own methods for computation instead of teaching them the traditional standard algorithms for the four arithmetic operations. Research on this issue is described, especially a project following pupils from their second to their fifth school year. In this, the pupils were not taught the standard algorithms at all, they had to resort to inventing their own methods for all computations, and these methods were discussed in groups or in the whole class. The article ends with a discussion of pros and cons of the ideas that are put forward.*
Keywords: -

1. Introduction

A lot of calculation today is carried out by using electronic means of computation, with calculators and computers. Besides, the pedagogical disadvantages of the traditional written algorithms for the four arithmetic operations have been emphasised by researchers in mathematics education for a long time (see e. g. Plunkett, 1979). In my opinion these two facts have not been taken into consideration in the mathematics classrooms, at least not in my own country. It is high time to ponder about what kind of knowledge of mathematics is important in our present society and will be important in the society of tomorrow. It is likely that the drill of traditional algorithms for the four arithmetic operations, which is all too common in today’s elementary schools, should be dismissed or at least heavily restricted and replaced by the pupils’ invention of their own methods for computation.

I started a project in one class in their second school year, and the project went on until the pupils had finished their fifth year, i. e. in the spring term 1998. In this project, the pupils have not been taught the traditional algorithms for the four arithmetic operations. Instead they have always been encouraged to find their own methods. The pupils have used mental computation as far as possible and written down notes to help them when the computations have been too complicated for the pupils to keep the results in their heads. This latter kind of computation I will call written computation, although we did not make use of the standard algorithms.

2. Background

There were three reasons for starting this project:

1. The electronic devices for computation already mentioned;
2. An increasing demand for a citizen's number sense (numeracy);
3. Social constructivism as a philosophy of learning.

I will discuss these three points very briefly.

1. I think that when computation is carried out by calculators and computers, it is still important that we ourselves understand the meaning of the computation and are able to check that a result is correct. We must therefore possess understanding and knowledge of numbers and relationships between numbers and the meaning of the different arithmetic operations as well as skill in mental computation and estimation. It has been pointed out that the methods for pencil-and-paper computations, that the pupils invent themselves, are much more like effective methods for mental arithmetic and computational estimation than the standard algorithms are.
2. To be able to do mental computation and estimation, a person needs good comprehension and understanding of numbers and relationships between numbers (number sense). There are many aspects of number sense, but I will only mention a few here that in my opinion are most essential for the issue under discussion. (See e. g. Reys, 1991.) A pupil with good number sense:

- understands the meanings and magnitudes of numbers;
- understands that numbers can be represented in different ways;
- knows the divisibility of numbers;
- knows how to use the properties of the arithmetic operations.

3. Social constructivism has been discussed quite a lot lately, and I do not want to give another contribution to this discussion. (See e. g. Cobb, 1997; Ernest, 1991; Ernest 1994.) It is, however, very difficult for me to see that the teaching of ready-made mechanical rules for computation is in accordance with social constructivism. Research has also shown that the traditional drill of standard algorithms has not been very successful (Narode, Board & Davenport, 1993). In my opinion, letting pupils invent and discuss with each other and with their teacher their own methods for computation would better adhere to the ideas of social constructivism. A teacher should feel free to show methods that s/he has found effective (including the standard algorithms, when her/his pupils are ready to understand them), but s/he should never force a standard method on everybody.

3. Previous research

In the CAN-project (Calculator Aware Number) in Britain (Duffin, 1996) the children, beside using their own methods for written computation, always had a calculator available, which they could use whenever they liked. Exploration and investigation of "how numbers work" was always encouraged, and the importance of mental arithmetic stressed.

One of the reported advantages of the CAN-project was that the teachers' style became less interventionist. The teachers began "to see the need to listen to and observe children's behaviour in order to understand the ways in which they learn". (Shuard et al, 1991, p. 56.) The teachers also recognised that the calculator "was a resource for generating mathematics; it could be used to introduce and develop mathematical ideas and processes". (Ibid p. 57.)

Kamii (1985, 1989, 1994) worked together with the children's class teachers in grades 1 - 3 in a similar way in the U. S. She did not teach the traditional algorithms but encouraged the children to invent their own methods for the four arithmetic operations. She also devoted much time to different kinds of mathematical games.

According to Kamii et al "many of the children who use the algorithm unlearn place value …" (Kamii, Lewis & Livingston, 1993/94, p. 202). They give

$$\begin{array}{r} 987 \\ \underline{+\ 654} \end{array}$$

as an example and compare pupils, who use their own methods with those using the algorithm. The former start with the hundreds and say: " 'Nine hundred and six hundred is one thousand five hundred. Eighty and fifty is a hundred thirty; so that's one thousand six hundred thirty …' ". The latter "unlearn place value by saying, for example, 'Seven and four is eleven. Put one down and one up. One and eight and five is fourteen. Put four down and one up. …' ". They state that children, when working with algorithms, have a tendency to think about every column as ones, and therefore the algorithm rather weakens than reinforces their understanding of place value. (Ibid p. 202.)

It is also interesting to follow the research carried out by Murray, Olivier, and Human in South Africa (e. g. Murray, Olivier & Human, 1994; Vermeulen Olivier & Human, 1996). Like the researchers mentioned above, they had their pupils invent their own strategies for computation, and above all they discussed strategies used for multiplication and division. In a summary of the results of their problem centred learning they state among other things:

> … students operate at the levels at which *they* feel comfortable. When a student transforms the given task into other equivalent tasks, these equivalent tasks are chosen because the particular student finds these tasks more convenient to execute.
> (Murray, Olivier & Human, 1994, p. 405.)

Narode, Board and Davenport (1993) concentrated on the negative role of algorithms for the children's understanding of numbers. In their research with first, second and third graders the researchers found out that after the children had been taught the traditional algorithms for addition and subtraction, they discarded their own invented methods, which they had used quite successfully before the instruction. The children also tried to use traditional algorithms in mental arithmetic, they gave many examples of misconceptions concerning place value, and they were all too willing to accept unreasonable results achieved by the wrong application of the traditional algorithms.

4. My own research

In my own research I wanted to see what changes will occur in a class when the children are given the possibility to invent and develop their own methods for written computation. In particular, I wanted to try to get answers to the following questions:

1. How is the pupils' number sense affected?
2. How is the pupils' ability to do mental computation and estimation affected?
3. How is the pupils' motivation for mathematics affected?
4. Is there a difference between girls' and boys' number sense and ability in mental computation and estimation?
5. Is there a difference between girls' and boys' motivation for mathematics?

I followed an ordinary Swedish class from their second school year up to and including their fifth. The data collection was finished at the end of the spring term 1998. In short the following steps were taken in the experimental class:

1. The children were encouraged and trained to use *other paper-and-pencil methods* rather than the traditional algorithms to carry out computations that they could not do mentally. No special methods were taught or forced upon the children. The methods were discussed in groups and in the whole class. The children's parents were also encouraged to help their children to use alternative computational methods and not to teach them the standard algorithms.

2. *Mental arithmetic and estimation* were encouraged and practised. The children were encouraged to invent their own methods, which were discussed in class.
3. The children had *calculators* in their desks. They were used for number experiments, for more complicated computations, and for checking computations made in other ways.
4. With the exceptions mentioned in points 1 - 3, the children followed a *traditional course*. The ordinary teacher had full responsibility for the mathematics periods. My own task was to design the project, to encourage and give advice to the teacher, and to evaluate the project.

Although the calculator could be said to be one of the reasons for the realisation of the experiment, it was not itself a major issue in it. However, I chose to let the pupils use calculators on some occasions, as it would have been illogical to pretend that they do not exist or that they are a resource that should only be used outside the classroom.

For the evaluation I used mainly qualitative methods:

- Clinical interviews,
- Observations,
- Copies of pupils' writing on the observed occasions,
- Interviews with pupils,
- Interviews with the teacher and weekly phone calls to him.

As the results stated below are only taken from observations, I will concentrate on them here. I undertook the observations when the pupils were working in small groups. In this way it was possible for me to follow the interaction in the groups. I used a tape-recorder during these sessions and made copies of the pupils' written work. Every pupil of the class was observed in this way at least twice per school year.

5. Some results

In this section I will give a few examples of the pupils' methods of computation. I will state if a computation was done only mentally or if some parts of the computation was written. However, I think there was very little difference between a computation that a pupil made in her/his mind and one where s/he made some notes to help her/him remember some intermediate results. I always interviewed the pupils about their solutions. I have thus been able to note the pupils' way of reasoning, even when the exercises were solved mentally. However, in the following text the mental computation, too, is written in mathematical symbols. I see no reason to translate the pupils' words into English, as part of the their thoughts and intentions would get lost anyhow.

A typical solution to an addition exercise with two three-digit numbers, 238 + 177, was: 200 + 100 = 300; 30 + 70 = 100; 7 + 8 = 15; 238 + 177 = 415. (This solution was written.)

A more special solution in addition looked like this: 157 + 66 = 160 + 63; 60 + 60 = 120; 100 + 120 + 3 = 223. (On the paper the boy only wrote 160 + 63 = 223. Thus, the solution was mainly done mentally.)

Several different methods were used in subtraction. This is one example for 147 - 58: 40 - 50 = - 10; 100 - 10 = 90; 7 - 8 = - 1; 90 - 1 = 89. (The solution was written.)

Another pupil solved the same exercise in the following way: 140 - 50 = 90; (90 + 7 = 97); 97 - 8 = 89. (Mental arithmetic.) The boy explained that he could not do the sum 7 - 8 but he managed 97 - 8.

I give a third example in subtraction, where a pupil uses even hundreds and tens. A boy computed 514 - 237 in this way: 500 - 200 = 300; 300 - 30 = 270; 270 - 7 = 263; 263 + 14 = 277. (Mental arithmetic.)

However, many pupils made mistakes in subtraction, because they mixed up numbers from the first and from the second term. E. g., two girls computed 514 - 237 as

500 - 200 = 300; 10 - 30 = 20; 4 - 7 = 3, and the answer was 323. (The solutions was written.)

A similar misunderstanding: 53 - 27: 50 - 20 = 30; 3 - 7 = 0. The answer was 30. (Mental arithmetic.)

I will give two examples in multiplication. In the first, the distributive property is used, in the second repeated addition.

7 x 320: 7 x 3 = 21; 2100; 7 x 2 = 14; 140. (The boy explained why he added one and two zeros respectively.); 2100 + 140 = 2240. (He only wrote the product 2100, the rest was done mentally.)

6 x 27: 27 + 27 = 54; 54 + 54 = 108; 108 + 50 = 158; 158 + 4 = 162. (The boy only wrote the number 54 and the final answer, the rest war done mentally.)

Finally, I turn to division. Again, I will give two examples, one, where the pupil partitioned the numerator in a sum of two terms and tried to divide one term at a time, and one, where the pupil guessed the quotient more or less intelligently and then tested its correctness with multiplication or addition.

236 ÷ 4: The girl first wrote 200 ÷ 4 = 50; 30 ÷ 4 + 6 ÷ 4. She then altered her writing to 36 ÷ 4 but made a mistake and got 19. After trying first 8 and then 9 she got the correct answer.

236 ÷ 4: The girl proceeded by trial and error. She found the number 50 pretty soon but had some trouble with the units. After trying 7 and 8, she found the number 9. She wrote:

	4	236	
9	9	9	9
50	50	50	50

and told me the answer 59.

As the girl in the first example of division, many pupils tried to partition the numerator into hundreds, tens and units, a method which they had used successfully in the three other arithmetic operations.

6. Discussion

6.1 Introduction

From these examples we can see that the pupils of the experimental class understood place value and were able to use it by partitioning in hundreds, tens and units. They could also use other ways to partition numbers, when these were more convenient. In the exercises they clearly showed their mastery of the following aspect of number sense: “Understands that numbers can be represented in different ways”.

The pupils also gave many examples of their mastery of the properties of the four arithmetic operations. They sometimes used a kind of compensation in addition and also subtraction to simplify the computations. In addition, they used the distributive property for multiplication and division over addition.

Many pupils solved the exercises mentally with few or no intermediate results written. Even the pupils who wrote very detailed notes used strategies that were very similar to those that are used in mental arithmetic.

However, we have to consider the gains and losses with an instruction, where the pupils are allowed to use their own methods for computation and the traditional algorithms are not taught. I will therefore give my own opinion of the advantages of pupils’ own methods and traditional algorithms respectively.

6.2 Advantages of the pupils’ own methods of computation.

- When the pupils get the chance to develop their methods themselves, these will in some sense be “the pupils’ property”.

- The methods are more like methods for mental arithmetic and computational estimation.
- As in mental arithmetic, the pupils almost always start computing from the left. By considering this position first, the pupils might get a sense of the magnitude of the result. We can also compare with computational estimation, where it might be enough to calculate with the numbers formed by the far left positions with a sidelong glance at the other digits.
- It is more natural to start reading from the left.
- The pupils practise their number sense when they are working in this way. They can clearly see what happens to the hundreds, the tens etc.
- It is easier to understand what happens in the computation. Thereby, the risk that the pupils will misunderstand the method and make systematic errors or forget what to do will be reduced.
- This way of working is in accordance with social constructivism.

However, in connection with the fourth statement I want to point out that there are also algorithms for all the four arithmetic operations, where one starts from the left, although they do not seem to be very common, except in division.

6.3 Advantages of the traditional standard algorithms.

- They have been invented and refined through centuries.Today, they are therefore very effective methods of computation.
- They can be used in about the same way, no matter how complicated the numbers involved are. If the computations are very laborious - e. g. multiplication of two three digit numbers or division by a two digit denominator - they are probably the only way, if one can only use paper and pencil.
- They are a part of the history of mathematics and are thus a cultural treasure that we should be careful with.

7. A final word

To be fair, I have to add that a non-standard method of computation can also be an algorithm in a negative sense - a plan that the pupil follows without being aware of what s/he is doing. However, as long as the pupil has invented her/his method herself/himself, there is no risk, but if s/he has taken over the method from a class mate or from the teacher without really understanding what s/he is doing, the risk is present.

As I see it, every pupil should start his learning of computation by inventing and using his own methods. We must look at computation as a process, where the pupil has to be creative and inventive, and from which the pupil can learn something.

However, the question whether we should teach the algorithms at all and, if so, when it should be done, remains. One extreme is not to teach them, because they are not needed in the society of today. When the computations are so complicated that we cannot use non-standard methods, we can turn to calculators or computers. In the other extreme, we introduce the standard algorithms pretty soon after the pupils have started developing their own methods. After that the pupils might be allowed to choose their methods at will.

Personally, I doubt if it is necessary to teach the standard algorithms at all. If teachers and pupils (or pupils' parents) insist, the teaching of them should be postponed to perhaps the sixth or seventh school year. By then, the pupils have, hopefully, already acquired good number sense, and therefore the teaching of the algorithms will not do any harm.

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