RELATION FUNCTION/AL ALGEBRA: AN EXAMPLE IN HIGH SCHOOL (AGE 15-16)

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Abstract: *In this proposal, we present some elements of a didactical engineering concerning an algebraic question from a double algebraic and topological viewpoint. Teachers know that " the study of sign of polynomials" is difficult in spite of the simplicity to build and use the array of signs. We think that a reason of this difficulty is in the following fact : impossibility for pupils to implement personal knowledge which is not algebraic but which can be pertinent to tackle the question. For instance, the intermediate value principle is a result which is very easily available as an implicit tool to draw graphics and exploit them.According to our analysis, the pupils have to deal with problems where functions, graphics and algebraic expressions are put in stage dialectically.*

Keywords: *functions, algebra, graphics, interplay between frameworks*

1. Problematic and methodology of the research

The research is concerned with the learning of polynomial fonctions for pupils 15-16 years old. We examine the conditions for a teaching based on polynomial functions in order to develop the knowledge of the pupils about such functions in the algebraic setting, and also to make easier conceptualization of and dealing with other functions : homographic, trigonometric functions.

Our aim is to study the consequences of a topological approach of algebraic questions on the knowledge of the pupils. We use a method of Didactical Engineering.

1. Choose a teaching object in the current program
2. Place the mathematical context in relation with the teaching tradition
3. Bring out hypotheses about the difficulties of the pupils and set the basis for a didactical engineering

4. Develop such an engineering, proceed to the a-priori analysis
5. Implement it and make an a-posteriori analysis of the collected data
6. Reproduce the implementation, under experimental control, after possible modification in view of the previous analysis
7. Test the supposedly acquired knowledge of the pupils in questions for which they are adapted tools
8. Compare the output of the pupils and their skill with expectations, and conclude about the relevance of the didactical hypotheses.

We present briefly below some of the issues listed above. This is work in progress.

2. Successive steps of the engineering

2.1 Study of polynomial functions (age 15-16)

The study of sign of polynomials, viewed as real functions of one real variable, is part of the curriculum in high school at age 15-16. It comes after the study of computations with algebraic expressions in the previous years. In particular the pupils have learned to expand expressions given as product of factors, and also in some "very simple" cases to transform expanded expressions into products of factors.

From a mathematical viewpoint, one form or the other is preferable according to the question dealt with. For instance, it is easy to determine the sign of a product of factors on an interval where each factor has a constant sign : one just has to know these signs, and the multiplication rule for signs. This study can be performed in a purely algorithmic way, ignoring the continuity aspect of the question. But is this really an advantage ?

2.2 The teaching tradition

From the teacher's viewpoint, it is customary to familiarize the pupils (age <15) with conversions of expressions, regardless of their mathematical relevance. At age 15-16, they are trained to form arrays of signs, a representation of data which allows an economical algebraic treatment. The method is, given a polynomial expressed as a product of factors of degree 1 or 2, to look for the zeroes, to determine the sign of each factor on each interval, and then apply the rule of signs to get the sign of the product. It is then efficient to organize information in an array.

However the economical feature is not an issue of the teaching. Though the mathematical object to be studied is *polynomial functions*, the stress is put on *polynomial* and very little on *functions*. The prevalent idea is that the teaching of algorithms will enable the pupils, in due time, to use such techniques properly, in situations which are possibly new for them, under their responsibility and control. It would be so for the study of functions, in which the sign of the derivative plays an important role.

On the pupils side, the availability of such algebraic tools is in fact not that clear. Low cost and efficiency are not perceived. Often you hear the pupils one year older saying: “we have been taught the array of signs last year, but we understood nothing” or “I have known how to do, but I forgot”. Moreover, even if they have good will, they are not always in a context of direct application. Here for instance, what can they do if they are given a polynomial function in an unusual way ? Have they a mean to test the validity of their possible results ? Would that be legitimate anyway, as the most widespread conviction is that the teacher is in charge of such controls.

2.3 An epistemological difficulty

Putting aside the problems of contract, the uneasiness of pupils in the algebraic study of polynomial functions may have several reasons. According to us, part of this uneasiness is due to the fact that the function aspect is conceptually underestimated or ignored, behind more algebraic features : number and values of zeroes, viewed in an algorithmic way. The situation in the neibourhood of a point where the function vanishes is not considered. The fact that the sign is constant between two consecutive zeroes is admitted without justification nor discussion. This has consequences on the signification given by pupils to the array of signs, and consequently to its use in contexts different from those who allowed introduction.

2.4 A didactical hypothesis

A hypothesis we are working on is the following : in order to be able to master easily the polynomial functions, the pupils need to conceive them both from an algebraic and topological viewpoint, in spite of the fact that the topological approach is not strictly necessary for the algebraic study.

In terms of teaching, the expression is that the study of zeroes and their multiplicity must be performed in relation with properties of continuity and derivability, at least implicitly. In this view, the cartesian graphical representation can play an important role as a tool. This study must be an issue in the learning of such functions.

2.5 The role of the cartesian graphical representation

This representation plays a multiple role :

- *in the conceptualization of functions, and in particular of polynomial functions.* It allows one to *see* a set of pairs of numbers (x,f(x)) coming from a computational program performed or evoked, as one new mathematical object. Many mathematical questions can be asked to specify the object. The study of these various questions will probably lead to get out of the framework in which the initial problem is given.
- *in the search of zeroes* For a polynomial function $x \to f(x)$, a value of x for which $f(x) = 0$ is easily perceived as a frontier between the values for which $f(x) > 0$ and those for which $f(x)<0$ after priming the process by numerical computation if necessary. Such a situation induces the pupils to use the intermediate value principle, a result which is very easily available as an implicit tool.
- *in the notion of multiplicity of roots* As soon as the pupils are able to conceive that a polynomial function is represented by a curve, the way the curve meets the horizontal axis, transversally or tangentially, more or less pressed, expresses the multiplicity of the root.

- *in the determination of signs*
  With the usual representation convention (positive above the x axis, negative below), the sign of the function on a given interval can be read on the graphic. Beyond this, reasoning has to take place. One also has to reason in order to check the validity of graphical information.

The graphic enables one to organize the interplay between frameworks, each one being used as a heuristic tool and mean of control for the other.

2.6 A didactical engineering based on frameworks interplay: algebraic-graphical-functions

Starting from the previous analysis and our didactical hypothesis, we propose to the pupils 3 types of problems, with different aims. The concept of function is the leading strand for the graphical/algebraic interplay. The issue is:

1. to realize that the *developed* and the *factorized* expressions allow an easy treatment of *different* mathematical questions.
2. to get acquainted with graphic as a tool for research, solution and control of results. Develop graphical/numerical interplay with graphic as the starting framework. Approach graphically the question of the sign of a polynomial and the relation between sign and the multiplicity of roots. In that phase of the work, the graphical medium is paper.
3. to work out algebraically on polynomials of various degree (up to 6 or 7) the role of zeroes and their multiplicity (in particular its parity) in the sign of polynomials, use the graphic as a tool to progress in the search of the sign. The medium can be paper or graphical calculator, these two being used in alternation.

2.7 Statements: choices and reasons for the choices

The sign of a polynomial function f depends on its roots and their multiplicity. Number, value and multipliciy of the roots are variables the teacher can play with.

The study relies on *continuity*, which is used as an *implicit tool*. Graphic provides an approach to this issue and leads to admit explicitly the *principle of intermediate values*.

Choice of the variables

- If the aim is the teaching of arrays of signs as an economical and efficient way of representing the relevant data on f in order to determine its sign, there should be several roots, but not too many so that the pupils can deal with the problem.
- If it is to establish the idea that f changes sign when x crosses a zero, but not always, then it is preferable to choose functions which display the two cases : simple and double roots.

Choice of functions

- One should choose a polynomial of degree at least 3 , preferably 4 or 5 , but not much more. As the attention is to be drawn on the unavoidable character of the study of roots, one shall choose to have f expressed as a product of factors of degree 1, possibly degree 2 with no real roots. Coefficients are chosen to be small integers, so as not to mislead the pupils by technical work.

Choice of the statement

It is worked out with reference with the theory of *Tool-Object Dialectic* and *Interplay between frameworks*.

- In a first time, the pupils can use their old knowledge, here compute the value f(x) corresponding to various choices of x .
- But their knowledge is not sufficient to solve completely the problem, here to give the sign of f(x) for any value of x assigned by the teacher. They would have to know the sign of f(x) for all values of x, i. e. for infinitely many values. The graphical representation acts as a lever to guide the research, in particular to determine the sign of f at the neighbourhood of a zero of the function, or between two consecutive zeroes. It should allow to formulate conjectures to be studied algebraically, for instance on the conditions under which the sign changes.
- An exposition of the results (some pupils, in turn, expose for the rest of the class the state of the work of their team) is an opportunity to make explicit what has

been solved and what remains open. It is also an opportunity to ask questions arising from the first studies (how many zeroes for f). To come back on known notions in new contexts, for instance on what is a function, on how to find the sign of a product of numbers when the sign of each one is known. It is an opportunity to organize in an array the information on the sign of the various factors which are involved in the expression of f .

- The analysis of the different examples should lead to an institutionnalization of the array of signs as a method.
- Familiarization with arrays of signs is planned, through the study of various examples.
- The study of well chosen non polynomial functions should allow to test the ability to reinvest as a tool what has been learned with polynomial functions.

2.8 Statement given in 2 times

1. *Calculators forbidden*

Giving x numerical values, you will get numerical values for the following expression:

$$f(x) = (x - 2) (2x - 3) (x + 5) (4x + 1) (1 - x)$$

Are they always positive ? Are they always negative question
Are they sometime positive, sometime negative, sometime zero ? Compute.
When you have an answer, call your teacher.

2. *Calculators forbidden*

$$f(x) = (x - 2) (2x - 3) (x + 5) (4x + 1) (1 - x)$$

Find a way which enables you to tell, very fast and reliably, when your teacher gives you a numerical value for x , whether the expression is > 0, < 0 or $= 0$.
Orders : *Only one answer accepted. Computing the expression is not allowed : it is too long.*
When you think you have a method, call your teacher.

2.9 Organization of the class and unfolding

At the end of the first session, the assessment brings out the difficulties of the pupils with the second question. Pupils repeating the class have looked for the places where f vanishes, and applied a principle of alternating sign each time the function passes by a zero. But they are unable to satisfy the requests for checking or justifying from their schoolmates who think this method just falls from heaven. Some pupils tried to use a calculator "just to see if it gives ideas", but they ran into trouble : with the window they use, the graph is in several pieces, and they don't know how to match the graphical information with the algebraic study. In particular they are not sure to have found all the vanishing values for f .

In the exposition of the results, a conviction graphically conforted is expressed : when the function passes from a positive value to a negative one (or the other way around), it has to go through the value 0. So between the two values, there is a value of x where f vanishes.

Hence a *method*: Look for the vanishing values. But a method which gives all of them.

A *question*: each time f vanishes, it changes sign. True or false ?

In the implementation, in one class this study has taken a long time. In another one where the rythm was much faster, the mere recording of cases where the statement was valid and others where it was not, was enough to focus interest on the reason for the difference in the behaviour of the functions.

We have decided to ask a problem where the issue was the study of the conditions of validity for the *conjecture: whenever* f *vanishes, it changes sign*.

For that study, we use an *interplay between functions and graphics.*

2.10 Statement 2

The conjecture was coming by graphical observation. In order to let the graphical representation play completely its heuristic function and its control function too, we have chosen a problem in which the *objects concerned are functions, but given by their graph* in cartesian frame (not by an equation). *The expected answer is graphical*. But in order to get it, one has to *work numerically* on the given functions.

Functions are given by their graph D_1 , D_2 , D_3 , D_4 (see Fig. 1 and Fig. 2). In each case, one has to consider two of them representing two functions f and g.

1. Determine the sign of the product h of the functions f and g .
   Definition of h: for each value of x , h(x) = f(x).g(x)
2. Propose a reasonable drawing of the product function h .
3. One of the lines being fixed, can one move the other one parallelly to itself in such a way that the product h has a constant sign when x varies question
   No measure with a ruler.

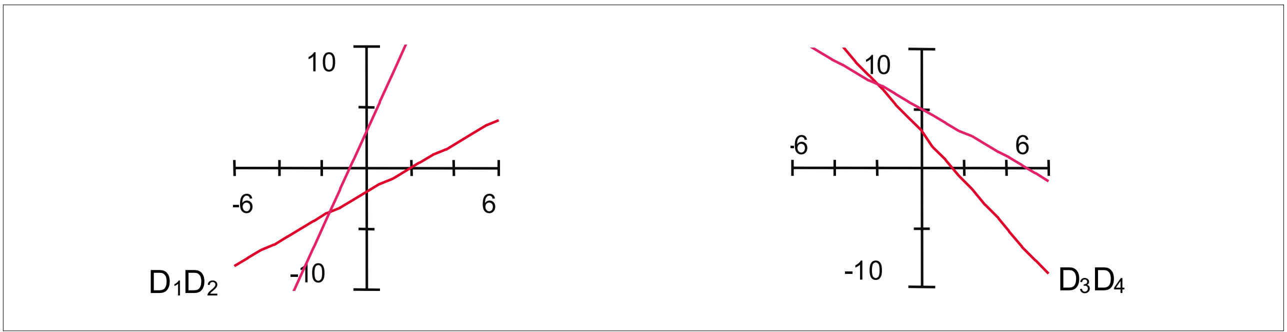

Fig. 1

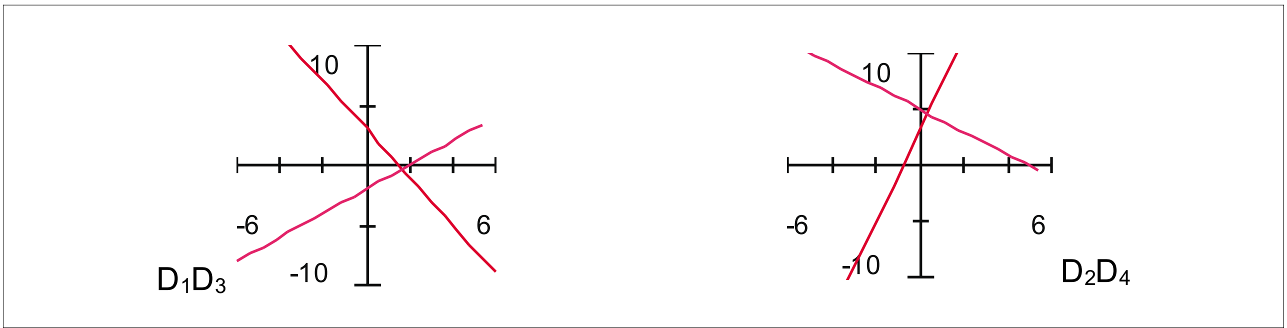

Fig. 2

Several variables are at the disposal of the teacher and are involved in the choice of the statement and the choice of the lines. We have chosen straight lines in order to be able to get without measuring 6 distinct certain points in the unknown graphic: those corresponding to $f(x) = 0$, 1 or -1 , and similarly for g . We have chosen the slopes : two examples with the same sign and two examples of opposite sign.

2.11 Realization and comments

During the implementation of the study, one observes that the pupils are bewildered by a practically "mute" graphic. In their practice, a function is given by its equation. How could they multiply graphics ?

It is clear that in this problem, the pupils have to distinguish between the object *function* and its representations. They have to operate on the object and give their answer in an assigned register. For this, the auxiliary working framework is the numerical part of the framework of functions. Only the use of the properties of the numbers 0 , 1 , -1 , and the relations between order and operations make it possible to make relevant selections and to get qualitative results consistent with the graphical data.

The pupils see no relation with the previous study on polynomial functions, their zero, their sign.

An intervention by the teacher unjams the situation and makes the research process start promptly. After recalling the definition of h , the teacher asks the question :
Are there points which are certain, points for which we are sure that they belong to the graph we look for?

The rumor on the interest of the zeroes of f and g, i. e. the points of intersection of the given lines with the x axis, sprays rapidly. In less than 10 minutes, several groups propose 4 certain points, and then those corresponding to $f(x) = -1$ and $g(x) = -1$ together with their geometrical construction.

The two other questions are also solved rapidly. The conclusions on which the pupils all agree are the following : we read on the graphic $f(x_1) = 0$, $g(x_2 = 0$, then

- $h(x_1) = 0$, $h(x_2) = 0$.

- If x_1 differs from x_2, the function h changes sign when x crosses x_1, and again when it crosses x_2.
- In order for h not to change sign, the interval $[x_1,x_2]$ must be reduced to a point. The functions f and g must vanish for the same value of x .

Their conclusion is still contextualized. But it is not far from a general statement which would have some meaning for them.

2.12 Conclusion, reinvestment

We have reproduced this engineering starting with the graphical problem, and later inserting in it a work which had been decided at national level by the commision inter IREM for computer science : *Draw a fish and send by internet a message without drawing to another class so that the pupils can draw the same fish, or one as much alike as possible*.

Students split their drawing in several parts and use functions to describe each of them. We have ended up with the study of the sign of a polynomial. The results show the process is very efficient.

The test performed on June confirms a good reinvestment of the array of signs in the study of the sign of polynomial functions, of homographic functions.

Familiarity with the graphic allows to propose graphical representations for the functions $\sin^2$, $\cos^2$, sin.cos with their period, sign and some symmetries, starting from the known graphical data on sin and cos . Then the question arises of whether the curves obtained are actually sine curves, and this leads after some algebraic-geometric work to conjecture trigonometric formulas which can later be proved.

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