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# TEACHING AND LEARNING LINEAR ALGEBRA IN FIRST YEAR OF FRENCH SCIENCE UNIVERSITY

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## 1. General presentation of the research

The teaching of linear algebra in France has undergone great modifications within the last thirty years. Today, linear algebra represents more or less a third of the mathematical contents taught in the first year of all French science universities. Traditionally, this teaching starts with the axiomatic definition of a vector space and finishes with the diagonalisation of linear operators. In a survey, Robert and Robinet (1989) showed that the main criticisms made by the students toward linear algebra concern the use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics. It is quite clear that many students have the feeling of landing on a new planet and are not able to find their way in this new world. The general attitude of teachers consists more often of a compromise: there is less and less emphasis on the most formal part of the teaching (especially at the beginning) and most of the evaluation deals with the algorithmic tasks connected with the reduction of matrices of linear operators. However, this leads to a contradiction which cannot satisfy us. Indeed, the students may be able to find the Jordan reduced form of an operator, but, on the other hand, suffer from severe misunderstanding on elementary notions such as linear dependence, generators, or complementary subspaces.

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In response to this situation, we have developed a research program on the learning and teaching of linear algebra in the first year of French science universities. This work, which started some ten years ago, includes the elaboration and evaluation of experimental teaching based on a substantial historical study and a theoretical approach within the French context of “didactique des mathématiques”, and in collaboration with other teams working on the teaching of linear algebra, in France and abroad (most of the results of these collaborations are gathered in Dorier 1997). At first, we made several analyses of students’ works in ordinary teaching; we also created tests and interviewed students and teachers in order to better understand the meaning of what we have called the “obstacle of formalism” (Dorier 1990 and 1991). We also investigated the history of linear algebra through the epistemological analysis of original texts from the 17th century up to recent developments (Dorier 1995a and 1997, Part 1). From this first stage of our research, we have drawn a first conclusion, which we will call the *fundamental epistemological hypothesis*:

- The theory of vector spaces is a very recent theory that emerged in the late 19th century but only spread in the 1930s. At this stage, it became widely used not so much because it allowed new problems to be solved (for instance linear functional equations were solved with use of the theory of determinants generalized to infinite dimensions) but mostly because it was a way of unifying different methods and tools used in different contexts and generalizing them (this was necessary because of the rapid increase in the number of new mathematical results). Therefore linear algebra is a unifying and generalizing theory. As a consequence, it is also a formal theory; it simplifies the solving of many problems, but the simplification is only visible to the specialist who can anticipate the advantage of generalization because he already knows many contexts in which the new theory can be used. For a beginner, on the other hand, the simplification is not so clear as the cost of learning many new definitions and theorems seems too great with regard to the use she or he can make of the new theory in contexts in which solving systems of linear equations is usually quite sufficient. In other words, the unifying and generalizing nature of linear algebra has a didactical consequence : it is difficult to motivate the learning of the new theory because its use will be profitable only after it may have been applied to a wide range of situations.

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In order to make the introduction of the theory of vector spaces more meaningful for students, we have developed a strategy based on two ideas :

- before the introduction of the formal theory, we make the students work on linear situations in three or four different contexts (linear systems, geometry, magic squares, equations of recurrence, ...). This phase is mainly experimental, even specific vocabulary is avoided if superfluous, but the results established are capitalized in a second phase, after the general theory has been introduced, in order to show the possible unification and generalization.
- we use what we called the “meta lever”. “Meta” means that a reflexive attitude from the student on his mathematical activity is expected, and “lever” points out something which has to be used at the right moment in the right place to help the student get into this reflexive attitude while achieving a mathematical task which has been prepared carefully. The general idea is to put the students in a mathematical activity that can be solved by him and from there, to make him analyze, in a reflexive attitude, some possibilities of generalization or unification of the methods he has developed by himself. Such a strategy induces flexibility in the means of access to knowledge. Therefore our teaching experiment pays much attention to changes in mathematical frameworks, semiotic registers of representation, languages or ways of thinking. Indeed, the historical analysis shows that linear algebra comes from very varied sources and that the interactions between different contexts and ways of expressing similar ideas was essential in its development.

On another hand, we have used our epistemological analysis of the historical context as a means of better understanding the mistakes of the students, not only through Bachelard’s epistemological obstacles but also in order to give more significance to the bare simplicity of formal modern definitions. The historical context provides us with possible ways of access to the knowledge. Yet, none is neither the best one, nor the shortest in terms of psychological cost, and if history is a privileged source of inspiration for building teaching situations, it is not sufficient by itself, the historical context has to be adapted to the didactical context which is usually very different. However, the historical source has to be as complete as possible (we do not trust second

hand summaries). In our work, we have explored in particular the role of the study of systems of linear equations and of geometry.

In this paper we cannot go into all the details of this work, but in order to make things more tangible, we would like to present one of the most recent part of our work on the concepts of linear dependence. It will put forward the specific use made of historical data in order to analyze students' misunderstanding and to elaborate didactical situations.

## **2. The case of linear dependence**

### **2.1 Introduction**

Linear dependence and independence, generators, basis, dimension and rank are the elementary concepts which constitute the foundations of the theory of vector spaces. For any mathematician, they appear to be very simple, clearly interrelated notions. Indeed, in the formal language of modern algebra they correspond to easily expressible definitions. Moreover, the logic of a hypothetico-deductive presentation induces a “natural” order between them (more or less the order given above) which reflects their intrinsic network of relations.

Let a system of linear equations have as many equations as unknowns ( $n$ ). A dependence between the equations of the system may be understood in different ways. If one is not familiar with the notion of linear dependence, but more familiar with the solving of the system, the dependence reflects an undetermination in the solutions of the system. Practically, it means that, in the process of resolution, one (or more) unknown(s) will be left undetermined. Therefore  $n$  dependent equations in  $n$  unknowns will be characterized by the fact that they determine less than  $n$  unknowns and therefore act as if they were less than  $n$ . With regard to the solving of equations, dependence is therefore an incident in the solving that results in the vanishing (within the process of solving) of at least one equation and the undetermination of at least one of the unknowns. It is an incident because  $n$  equations usually determine  $n$  unknowns exactly. If the method for solving the system uses linear combinations, this incident may be related to the fact that a linear combination of the equations is zero. If the dependence is

“obvious”, one may even see directly that one equation is a linear combination of the others, although this will not be the central characteristic of the dependence.

## 2.2 Historical background

This might be difficult to admit for a modern mathematician so familiar with the vocabulary and basic notions of linear algebra. But such a way of considering dependence between equations may be found (with more or less the same words) in a text from 1750, by Euler, and still prevailed in most of the texts about linear equations up to the end of the 19<sup>th</sup> century (see Dorier 1993, 1995, 1996). Euler’s text is the first one in which the question of dependence is discussed. The general idea that  $n$  equations determine  $n$  unknowns was so strong that nobody had bothered to discuss the odd case, until Euler was confronted with Cramer’s paradox and pointed out this particularity. 1750 is also the year Cramer published the treatise that introduced the use of determinants which was to dominate the study of linear equations until the first quarter of the 20<sup>th</sup> century. In this context, dependence was characterized by the vanishing of the determinant. The notion of linear dependence, now basic in modern linear algebra, did not appear in its modern form until 1875. Frobenius introduced it, pointing out the similarity with the same notion for  $n$ -tuples. He was therefore able to consider linear equations and  $n$ -tuples as identical objects with regard to linearity. This simple fact may not seem very relevant but it happened to be one of the main steps toward a complete understanding of the concept of rank. Indeed in the same text, Frobenius was able not only to define what we would call a basis of solutions but he also associated a system of equations to such a basis (each  $n$ -tuple is transformed into an equation). Then he showed that any basis of solutions of this associated system has an associated system with the same set of solutions as the initial system. This first result on duality in finite-dimensional vector spaces showed the double level of invariance connected to rank both for the system and for the set of solutions. Moreover, Frobenius’ approach allowed a system to be seen as part of a class of equivalent systems having the same set of solutions: a fundamental step toward the representation of sub-spaces by equations. This shows how adopting a formal definition (here of linear dependence and independence) may be a fundamental step in the construction of a theory, and is therefore an essential intrinsic constituent of this theory.

### 2.3 Didactical implications

Anyone who has taught a basic course in linear algebra knows how difficult it may be for a student to understand the formal definition of linear independence, and to apply it to various contexts. Moreover, once students have proved their ability to check whether a set of  $n$ -tuples, equations, polynomials or functions are independent, they may not be able to use the concept of linear independence in more formal contexts. Robert and Robinet (1989) have tested beginners on these two questions :

1. Let  $U, V$  and  $W$  be three vectors in  $\mathbb{R}^3$ , if they are two by two non collinear, are they independent question
- 2.1 Let  $U, V$  and  $W$  be three vectors in  $\mathbb{R}^3$ , and  $f$  a linear operator in  $\mathbb{R}^3$ , if  $U, V$  and  $W$  are independent, are  $f(U), f(V)$  and  $f(W)$  independent question
- 2.2 Let  $U, V$  and  $W$  be three vectors in  $\mathbb{R}^3$ , and  $f$  a linear operator in  $\mathbb{R}^3$ , if  $f(U), f(V)$  and  $f(W)$  are independent, are  $U, V$  and  $W$  independent question

These questions were generally failed by beginners. In the three cases, they used the formal definition of linear independence and tried different combinations with the hypotheses and the conclusions, so that, to the first question they answered yes, and to the last two they answered respectively yes and no, despite coming close to writing the correct proof for the correct answers. In their initial analysis, the authors concluded that the main difficulty was related with the use of the implication and a confusion between hypothesis and conclusion. This is indeed an obvious difficulty in the use of the formal definition of linear independence. A few years later, Dorier (1990) used the same questions. Previously, he had set up a test, to evaluate the students' ability in elementary logic and particularly in the use of implication. The result was surprisingly that the correlation between the questions on the use of implication and the three questions above was insignificant, in some cases it was even negative. Yet, on the whole, there was quite a good correlation between the two tests. This shows that if a certain level of ability in logic is necessary to understand the formalism of the theory of vector-space, general knowledge, rather than specific competence is needed. In other words, if some difficulties in linear algebra are due to formalism, they are specific to linear algebra and have to be overcome essentially in this context.

## 2.4 A proposition

In response to this analysis, we set up a teaching experiment based on the following scheme:

- after the definition of a vector space and sub-space and linear combination, we defined the notion of generator. A set of generator concentrates all the information we have on the sub-space, it is therefore interesting to reduce it to the minimum. Therefore, the question is to know when it is possible to take away one generator, the remaining vectors being still generators for the whole sub-space. The students easily found that the necessary and sufficient condition is that the vector to be taken away is a linear combination of the others. This provides the definition of linear dependence : “a vector is linearly dependent on others if and only if it is a linear combination of them”. This definition is very intuitive, yet it is not completely formal, and it needs to be generalized to sets of one vector. It induces without difficulty the definition of a set of independent vectors as a set of which no vector is a linear combination of the others. To feel the need for a more formal definition, one just has to reach the application of this definition. Indeed, suppose one has to answer the question: “are these vectors independent or not?”. With the definition above, one needs to look for each vector, one after the other, to see if it is a linear combination of the others. After a few examples, with at least three vectors, it is easy to explain to the students that it would be better to have a definition in which all the vectors play the same part (it is also interesting to insist on the fact that this is a general statement in mathematics). One is now ready to transform the definition of linear dependence into : “vectors are linearly independent if and only if there exists a zero linear combination of them, whose coefficients are not all zero.” The definition of linear independence being the negation of this, it is therefore a pure problem of logic to reach the formal definition of linear independence. A pure problem of logic, but in a precise context, where the concepts have made sense to the students previously.

This approach has been proved to be efficient with regards to the student’s ability to use the definitions of linear dependence and independence, even in formal contexts such as in the three questions quoted above.

Moreover, it is quite a discovery for the students to realize that a formal definition may be more practical than an “intuitive” one. Indeed, most of them keep seeing the fact that a vector is a linear combination of the others as a consequence of the definition of linear dependence. Therefore they believe that this consequence is the practical way of proving that vectors are or are not independent, even if that is against their use of these definitions.

## 2.5 Conclusions

This example is relevant with regard to the question about the role of formalism in linear algebra. Formalism is what students themselves confess to fear at most in the theory of vector spaces. One didactical solution is to avoid formalism as far as possible, or at least to make it appear as a final stage gradually. Because we think that formalism is essential in this theory (our historical analysis has confirmed this epistemological fact), we give a different answer: formalism must be put forward in relation to intuitive approaches as the means of understanding the fundamental role of unification and generalization of the theory (see Freudenthal 1983). This has to be an explicit goal of the teaching. This is not incompatible with a gradual approach toward formalism, but it induces a different way of thinking the previous stages. Formalism is not only the final stage in a gradual process in which objects become more and more general, it must appear as the only means of comprehending different previous aspects within the same language. The difficulty here is to give a functional aspect to formalism while approaching it more intuitively.

We have explained above that dependence for equations could not mean linear dependence, although it is logically equivalent to it. Moreover, in any context, dependence is very different whether it applies to fewer or more than two vectors. It is important to understand how dependence is the generalization to more than two vectors of the notions of collinearity or proportionality. Question 1, quoted above, is an example of the difficulty students may encounter in this matter. In geometry, dependence may have different aspects too. “Three vectors are dependent if they are on the same plane” is the most intuitive way of imagining the dependence. What is the relation between this point of view and the fact that one vector is a linear combination

of the others ? Students have a conception of coplanar points prior to their knowledge of linear algebra. Is this conception compatible with the notion of linear combination?

Linear dependence is a formal notion that unifies different types of dependences which interact with various previous intuitive conceptions. It has been shown above how in the historical development of linear algebra the understanding of this fact was essential for the construction of the concept of rank and partly of duality. In the teaching, this questioning has to be explicit, if we do not want misunderstanding to persist. Therefore even at the lowest levels of the theory the question of formalism has to be raised in interaction with various contexts where previous intuitive conceptions have been built by the students. The construction of a formal approach right from the beginning is a necessary condition to the understanding of the profound epistemological nature of the theory of vector spaces. In this sense, formalism has to be introduced as the answer to a problem that students are able to understand and to make their own, in relation to their previous knowledge in fields where linear algebra is relevant. These include at least geometry and linear equations but may also include polynomials or functions, although in these fields one may encounter more difficulties.

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